BIJU PATNAIK UNIVERSITY OF TECHNOLOGY, ODISHA

Lecture Notes

On

QUANTITATIVE TECHNIQUE-II

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- A stochastic process is a indexed collection of random variables $\{X_t\}$ = $\{ X_0, X_1, X_2, \ldots \}$ for describing the behavior of a system operating over some period of time.
- For example :
  - $X_0 = 3, X_1 = 2, X_2 = 1, X_3 = 0, X_4 = 3, X_5 = 1$
- An inventory example:
- A camera store stocks a particular model camera.
- D_t represents the demand for this camera during week t.
- D_t has a Poisson distribution with a mean of 1.
- X_t represents the number of cameras on hand at the end of week t. ($X_0 = 3$)
- If there are no cameras in stock on Saturday night, the store orders three cameras.
- $\{ X_t \}$ is a stochastic process.
- $X_{t+1} = \max\{ 3 - D_{t+1}, 0 \}$ if $X_t = 0$

  $\max\{ X_t - D_{t+1}, 0 \}$ if $X_t \geq 0$
- A stochastic process $\{X_t\}$ is a Markov chain if it has Markovian property.
- Markovian property:
  - $P\{ X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i \}$

  $= P\{ X_{t+1} = j \mid X_t = i \}$
- $P\{ X_{t+1} = j \mid X_t = i \}$ is called the transition probability.
- Stationary transition probability:
  - If ,for each i and j, $P\{ X_{t+1} = j \mid X_t = i \} = P\{ X_1 = j \mid X_0 = i \}$, for all t, then the transition probability are said to be stationary.
- Formulating the inventory example:
  - Transition matrix:

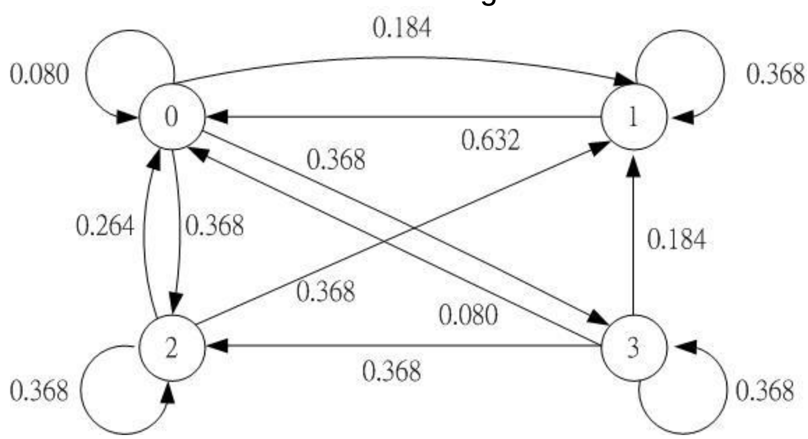
$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & p_{00} & p_{01} & p_{02} & p_{03} \\ 1 & p_{10} & p_{11} & p_{12} & p_{13} \\ 2 & p_{20} & p_{21} & p_{22} & p_{23} \\ 3 & p_{30} & p_{31} & p_{32} & p_{33} \end{array}$$

- $X_{t+1} = \max\{\ 3 - D_{t+1},\ 0\ \}$ if $X_t = 0$

  $\max\{\ X_t - D_{t+1},\ 0\ \}$ if $X_t \geq 1$
- $p_{03} = P\{\ D_{t+1} = 0\ \} = 0.368$
- $p_{02} = P\{\ D_{t+1} = 1\ \} = 0.368$
- $p_{01} = P\{\ D_{t+1} = 2\ \} = 0.184$
- $p_{00} = P\{\ D_{t+1} \geq 3\ \} = 0.080$

$$
\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}
$$

- The state transition diagram:



- n-step transition probability :
  - $p_{ij}^{(n)} = P\{\ X_{t+n} = j \mid X_t = i\ \}$
- n-step transition matrix :

$$
\mathbf{P}^{(n)} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & \dots & M \\
0 & P_{00}^{(n)} & P_{01}^{(n)} & \dots & P_{0M}^{(n)} \\
1 & P_{10}^{(n)} & P_{11}^{(n)} & \dots & P_{1M}^{(n)} \\
\vdots & \dots & \dots & \dots & \dots \\
M & P_{M0}^{(n)} & P_{M1}^{(n)} & \dots & P_{MM}^{(n)}
\end{array}
$$

- Chapman-Kolmogorove Equation :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for all i = 0, 1, …, M,
j = 0, 1, …, M,
and any m = 1, 2, …, n-1,
n = m+1, m+2, …

-
- The special cases of m = 1 leads to :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(1)} p_{kj}^{(n-1)} \qquad \text{for all i and j}$$

- Thus the n-step transition probability can be obtained from one-step transition probability recursively.
- Conclusion :
  - $\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}\mathbf{P}\mathbf{P}^{(n-2)} = \dots = \mathbf{P}^{n}$
- n-step transition matrix for the inventory example :

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}$$

$$\mathbf{P}^{(4)} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.289 & 0.286 & 0.261 & 0.164 \\ 1 & 0.282 & 0.285 & 0.268 & 0.166 \\ 2 & 0.284 & 0.283 & 0.263 & 0.171 \\ 3 & 0.289 & 0.286 & 0.261 & 0.164 \end{array}$$

- What is the probability that the camera store will have three cameras on hand 4 weeks after the inventory system began ?
- $P\{ X_n = j \} = P\{ X_0 = 0 \} p_{0j}^{(n)} + P\{ X_0 = 1 \} p_{1j}^{(n)} + \dots$

  $+ P\{ X_0 = M \} p_{Mj}^{(n)}$
- $P\{ X_4 = 3 \} = P\{ X_0 = 0 \} p_{03}^{(4)} + P\{ X_0 = 1 \} p_{13}^{(4)}$

  $+ P\{ X_0 = 2 \} p_{23}^{(4)} + P\{ X_0 = 3 \} p_{33}^{(4)}$

  $= (1)\, p_{33}^{(4)} = 0.164$
- Long-Run Properties of Markov Chain
  - Steady-State Probability

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}$$

$$
\mathbf{P}^{(8)} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & 2 & 3 \\
0 & 0.286 & 0.285 & 0.264 & 0.166 \\
1 & 0.286 & 0.285 & 0.264 & 0.166 \\
2 & 0.286 & 0.285 & 0.264 & 0.166 \\
3 & 0.286 & 0.285 & 0.264 & 0.166
\end{array}
$$

- The steady-state probability implies that there is a limiting probability that the system will be in each state j after a large number of transitions, and that this probability is independent of the initial state.
- Not all Markov chains have this property.

$$
\begin{array}{c|cccc}
\text{state} & 0 & 1 & 2 & 3 \\
0 & \pi_0 & \pi_1 & \pi_2 & \pi_3 \\
1 & \pi_0 & \pi_1 & \pi_2 & \pi_3 \\
2 & \pi_0 & \pi_1 & \pi_2 & \pi_3 \\
3 & \pi_0 & \pi_1 & \pi_2 & \pi_3
\end{array}
$$

- Steady-State Equations :

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij} \qquad \text{for } i = 0, 1, \ldots, M$$

$$\sum_{j=0}^{M} \pi_j = 1$$

- which consists of M+2 equations in M+1 unknowns.

- The inventory example :
- $\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30}$,
- $\pi_1 = \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31}$,
- $\pi_2 = \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32}$,
- $\pi_3 = \pi_0 p_{03} + \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33}$,
- $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$.

- $\pi_0 = 0.080\pi_0 + 0.632\pi_1 + 0.264\pi_2 + 0.080\pi_3$,
- $\pi_1 = 0.184\pi_0 + 0.368\pi_1 + 0.368\pi_2 + 0.184\pi_3$,
- $\pi_2 = 0.368\pi_0 + \quad + 0.368\pi_2 + 0.368\pi_3$,
- $\pi_3 = 0.368\pi_0 + \quad + \quad + 0.368\pi_3$,
- $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$.

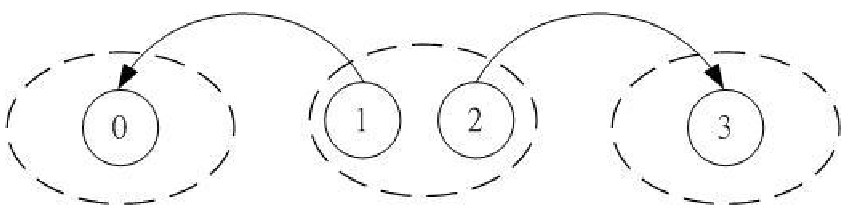
- $\pi_0 = 0.286, \pi_1 = 0.285, \pi_2 = 0.263, \pi_3 = 0.166$

Classification of States of a Markov Chain

- Accessible :
  - State j is accessible from state i if $P_{ij}^{(n)} > 0$ for some n ≥ 0.
- Communicate :
  - If state j is accessible from state i and state i is accessible from state j, then states i and j are said to communicate.
  - If state i communicates with state j and state j communicates with state k, then state j communicates with state k.
- Class :

- The state may be partitioned into one or more separate classes such that those states that communicate with each other are in the same class.
  - Irreducible :
    - A Markov chain is said to be irreducible if there is only one class, i.e., all the states communicate.
  - A gambling example :
    - Suppose that a player has $1 and with each play of the game wins $1 with probability p > 0 or loses $1 with probability 1-p. The game ends when the player either accumulates $3 or goes broke.

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 1-p & 0 & p & 0 \\ 2 & 0 & 1-p & 0 & p \\ 3 & 0 & 0 & 0 & 1 \end{array}$$



  - Transient state :
    - A state is said to be a transient state if, upon entering this state, the process may never return to this state. Therefore, state I is transient if and

only if there exists a state j (j≠i) that is accessible from state i but not vice versa.

- Recurrent state :
  - A state is said to be a recurrent state if, upon entering this state, the process definitely will return to this state again. Therefore, a state is recurrent if and only if it is not transient.
- Absorbing state :

- A state is said to be an absorbing state if, upon entering this state, the process never will leave this state again. Therefore, state i is an absorbing state if and only if P_{ii} = 1.

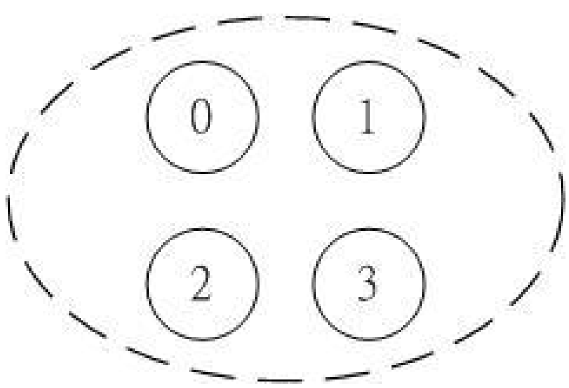
$$
\mathbf{P} = \begin{array}{ccccc}
\text{state} & 0 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1-p & 0 & p & 0 \\
2 & 0 & 1-p & 0 & p \\
3 & 0 & 0 & 0 & 1
\end{array}
$$

- Period :
  - The period of state i is defined to be the integer t (t>1) such that $P_{ii}^{(n)}$ = 0 for all value of n other than t, 2t, 3t, ...
  - $P_{11}^{(k+1)} = 0$, k = 0, 1 ,2 , ...
- Aperiodic :
  - If there are two consecutive numbers s and s+1 such that the process can be in the state i at

times s and s+1, the state is said to be have period 1 and is called an aperiodic state.

- Ergodic :
  - Recurrent states that are aperiodic are called ergodic states.
  - A Markov chain is said to be ergodic if all its states are ergodic.
- For any irreducible ergodic Markov chain, steady-state probability, $\lim_{n\to\infty} p_{ij}^{(n)}$,exists.
  - An inventory example :
- The process is irreducible and ergodic and therefore, has steady-state probability.

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}$$



- First Passage time :

- The first passage time from state i to state j is the number of transitions made by the process in going from state i to state j for the first time.

- Recurrence time :
  - When j = i, the first passage time is just the number of transitions until the process returns to the initial state i and called the recurrence time for state i.
- Example :
  - $X_0 = 3, X_1 = 2, X_2 = 1, X_3 = 0, X_4 = 3, X_5 = 1$
  - The first passage time from state 3 to state 1 is 2 weeks.
- The recurrence time for state 3 is 4 weeks.
  - $f_{ij}^{(n)}$ denotes the probability that the first passage time from state i to state j is n.
- Recursive relationship :

$$f_{ij}^{(n)} = \sum_{k \neq j} p_{ik} f_{kj}^{(n-1)} \qquad f_{ij}^{(1)} = p_{ij}^{(1)} = p_{ij} \qquad f_{ij}^{(2)} = \sum_{k \neq j} p_{ik} f_{kj}^{(1)}$$

- The inventory example :
  - $f_{30}^{(1)} = p_{30} = 0.080$
  - $f_{30}^{(2)} = p_{31} f_{10}^{(1)} + p_{32} f_{20}^{(1)} + p_{33} f_{30}^{(1)}$

    $= 0.184(0.632) + 0.368(0.264) + 0.368(0.080) = 0.243$
  - ... ...
- Sum :

$$\sum_{n=1}^{\infty} f_{ij}^{(n)} \leqq 1$$

- Expected first passage time :

- Expected first passage time :
- μ_{ij} =
-

- The inventory example :
  - $\mu_{30} = 1 + p_{31}\mu_{10} + p_{32}\mu_{20} + p_{33}\mu_{30}$
  - $\mu_{20} = 1 + p_{21}\mu_{10} + p_{22}\mu_{20} + p_{23}\mu_{30}$
  - $\mu_{10} = 1 + p_{11}\mu_{10} + p_{12}\mu_{20} + p_{13}\mu_{30}$

μ_{10} = 1.58 weeks, μ_{20} = 2.51 weeks, μ_{30} = 3.50 weeks

- Absorbing states :
  - A state k is called an absorbing state if $p_{kk} = 1$, so that once the chain visits k it remains there forever.
- An gambling example :
  - Suppose that two players (A and B), each having \$2, agree to keep playing the game and betting \$1 at a time until one player is broke. The probability of A winning a single bet is 1/3.
- The transition matrix form A's point of view

$$P = \begin{array}{c|ccccc} \text{state} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2/3 & 0 & 1/3 & 0 & 0 \\ 2 & 0 & 2/3 & 0 & 1/3 & 0 \\ 3 & 0 & 0 & 2/3 & 0 & 1/3 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

- Probability of absorption :
  - If k is an absorbing state, and the process starts in state i, the probability of ever going to state k is called the probability of absorption into state k, given the system started in state i.
- The gambling example :

$f_{20} = 4/5$, $f_{24} = 1/5$