BIJU PATNAIK UNIVERSITY OF TECHNOLOGY, ODISHA

Lecture Notes

On

STOCHASTIC PROCESSES

Prepared by,

Dr. Subhendu Kumar Rath,

BPUT, Odisha.

Stochastic Processes

Dr Subhendu Kumar Rath,

Dy. Registrar, BPUT

Stochastic process

X(t) is the state of the process (measurable characteristic of interest) at time t

- the state space of the a stochastic process is defined as the set of all possible values that the random variables *X(t)* can assume
- when the set *T* is countable, the stochastic process is a **discrete time process**; denote by *{Xn, n=0, 1, 2, …}*
- when *T* is an interval of the real line, the stochastic process is a continuous time process; denote by *{X(t), t≥0}*

Stochastic process

Hence,

- a stochastic process is a family of random variables that describes the evolution through time of some (physical) process.
- usually, the random variables $X(t)$ are dependent and hence the analysis of stochastic processes is very difficult.
- Discrete Time Markov Chains (**DTMC**) is a special type

of stochastic process that has a very simple dependence among $X(t)$ and renders nice results in the analysis of $\{X(t), t \in T\}$ under very mild assumptions.

Example of stochastic processes

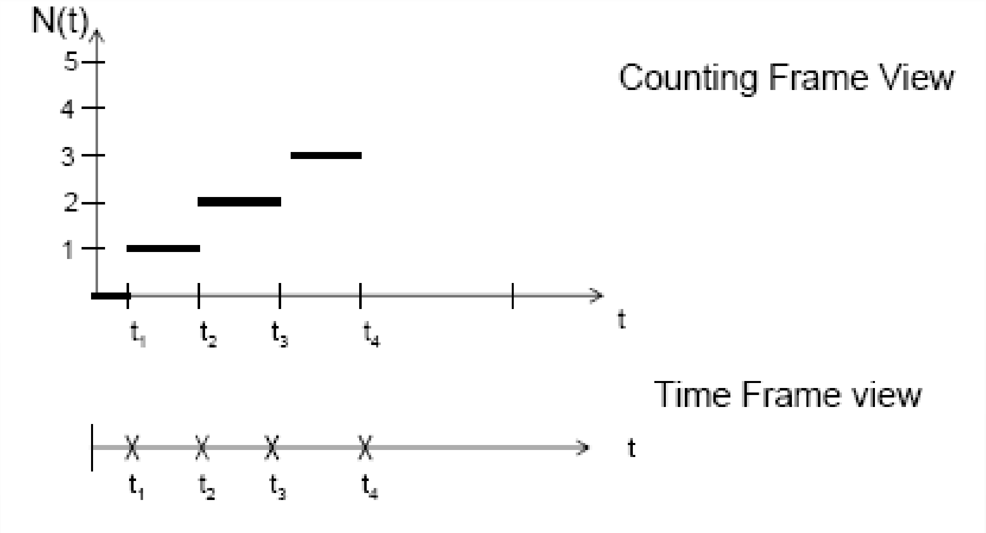
Refer to X(t) as the state of the process at time t

→ A stochastic process {X(t), t∈T} is a time indexed collection of random variables

- X(t) might equal the total number of customers that have entered a supermarket by time t
- X(t) might equal the number of customers in the supermarket at time t
- X(t) might equal the stock price of a company at time t

Counting process

- Definition:
  - A stochastic process {N(t), t≥0} is a counting process if N(t) represents the total number of “events” that have occurred up to time t



Counting process

- Examples:
  - If $N(t)$ equal the number of persons who have entered a particular store at or prior to time t, then $\{N(t), t \geq 0\}$ is a counting process in which an event corresponds to a person entering the store
    - If $N(t)$ equal the number of persons in the store at time t, then $\{N(t), t \geq 0\}$ would not be a counting process. Why?
  - If $N(t)$ equals the total number of people born by time t, then $\{N(t), t \geq 0\}$ is a counting process in which an event corresponds to a child is born
  - If $N(t)$ equals the number of goals that Ronaldo has scored by time t, then $\{N(t), t \geq 0\}$ is a counting process in which an event occurs whenever he scores a goal

Counting process

- A counting process N(t) must satisfy
  - $N(t) \geq 0$
  - N(t) is integer valued
  - If $s \leq t$, then $N(s) \leq N(t)$
  - For $s<t$, $N(t)-N(s)$ equals the number of events that have occurred in the interval (s,t), or the increments of the counting process in (s,t)
- A counting process has
  - **Independent increments** if the number of events which occur in disjoint time intervals are independent
  - **Stationary increments** if the distribution of the number of events which occur in any interval of time depends only on the length of the time interval

Independent increment

- This property says that numbers of events in disjoint intervals are independent random variables.

- Suppose that $t_1 < t_2 \leq t_3 < t_4$. Then $N(t_2)-N(t_1)$, the number of events occurring in $(t_1, t_2]$, is independent of $N(t_4)-N(t_3)$, the number of events occurring in $(t_3, t_4]$.

Example

Dependent increments:

- Suppose N(t) is the number of babies born by year t.

- If N(t) is very large, then it is probable that there are many people alive at time t; this would then lead us to believe that the number of new births between time t and t+s would also tend to be large.

- Hence $\{N(t), t \geq 0\}$ does not have independent increments.

Stationary increment

- This property states that the distribution of the number of events which occur in any interval of time depends only on the length of the time interval

- Suppose $t_1<t_2$, and $s>0$. Then
  - $N(t_2+s)-N(t_1+s)$, number of events in the interval (t_1+s, t_2+s), has the same distribution as
  - $N(t_2)-N(t_1)$, the number of events in interval (t_1, t_2).

Example

Non-stationary increments:

- The number of consumers who have entered the university canteen obviously does not have stationary increments. The arrival rates are higher during the lunch time

Poisson process (definition 1)

- The counting process {N(t), t≥0} is a **Poisson Process** with rate λ, λ>0, if
  1. N(0) = 0
  2. The process has independent increments
  3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, ∀s, t≥0,

$$P\{N(t+s)-N(s)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

- From condition 3, the Poisson process has stationary increments and E[N(t)]=λt.
- From definition 1, a Poisson process with rate λ means that at any t>0, the number of events follows a Poisson distribution with mean λt.

Example

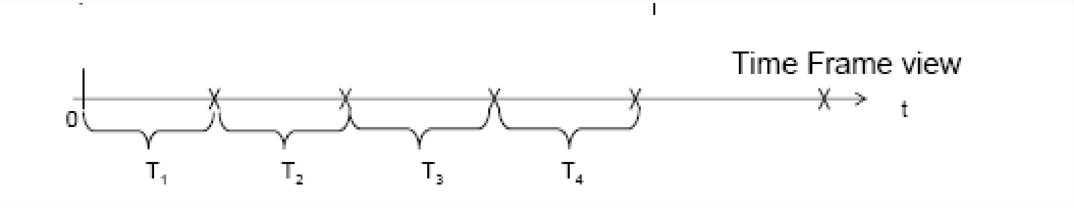
- The arrival of customers at a café is a Poisson process with rate 4 per minute. Find the probability that there is no less than 5 arrivals
  - a) between time (0,2] (the first two minutes)
  - b) between time (4,8].

Identifying a Poisson process

- To determine if an arbitrary counting process is a Poisson process, we need to show that conditions 1, 2, 3 are satisfied.
  - Condition 1: states that the counting of events begins at time 0
  - Condition 2: can usually be directly verified from our knowledge of the process
  - Condition 3: hard to determine

Inter-arrival time distribution

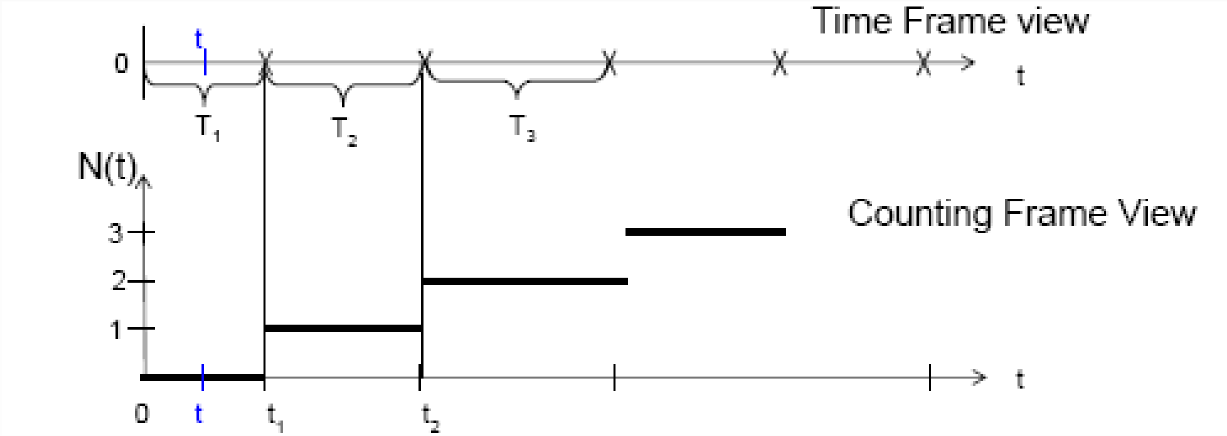
- For a Poisson process, denote T_1 as the time of the first event, and T_n as the time between the (n-1)st and nth event, for n>1.



- The sequence $\{T_n, n=1, 2, \ldots\}$ is called the sequence of interarrival times.
- What is the distribution of T_n?

Inter-arrival time distribution

- Note that the event $\{T_1>t\}$ takes place if and only if no events of the Poisson process occurs in the time interval [0,t], $\{T_1>t\} \leftrightarrow \{N(t)=0\}$



- Thus $P(T_1>t) = P(N(t)=0)=e^{-\lambda t}$
- Hence T_1 has exponential distribution with mean $1/\lambda$

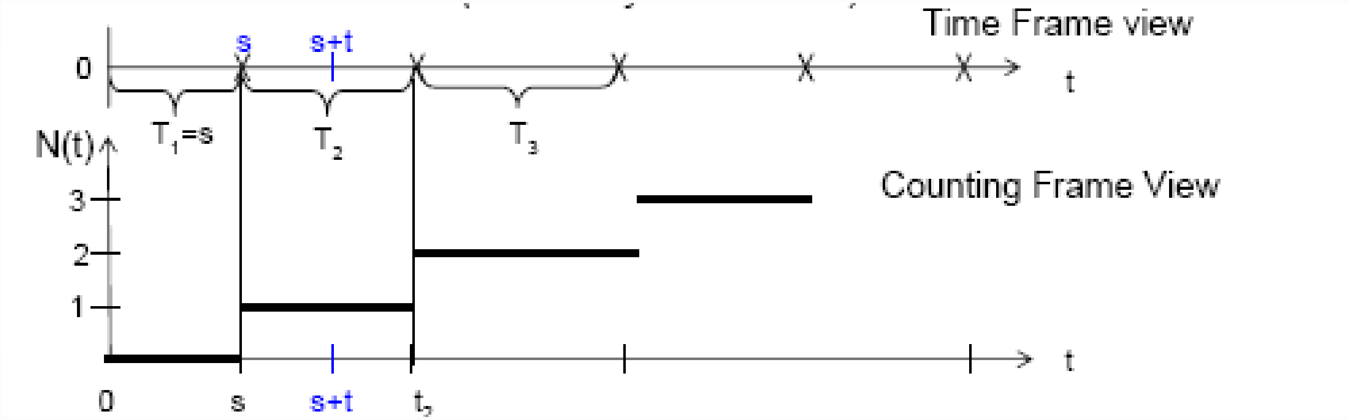
Inter-arrival time distribution

- To obtain distribution of T_2, condition on T_1

  $P\{T_2>t \mid T_1=s\} = P\{0 \text{ events in } (s,s+t] \mid T_1=s\}$

  $= P\{0 \text{ events in } (s, s+t]\}$ *(indept increments)*

  $= e^{-\lambda t}$ *(stationary increments)*



- Repeating the same argument yields the following **Proposition:**

  T_n, n=1, 2, … are independent identically distributed exponential random variables having mean $1/\lambda$

Inter-arrival times

Remark 1:

- This proposition is actually quite intuitive.
- The assumption of stationary and independent increments is basically equivalent to saying that at any point in time, the process *probabilistically* restarts itself.
  - That is, the process at any point on is independent of all that has previously occurred (by independent increments), and also has the same distribution as the original process (by stationary increments).
- In other words, the process has no memory, and hence the exponential interarrival times are expected.

Inter-arrival times

Remark 2:

- Another way to obtain the distribution of T_2 and so on, is to make use of the independent and stationary increments of the Poisson process. As such, the process *probabilistically* restarts itself at any point in time, so looking at the Time Frame view, the time origin can be 'pushed' forward, and T_2, T_3, … can be viewed similarly like T_1.

Inter-arrival times

Remark 3:

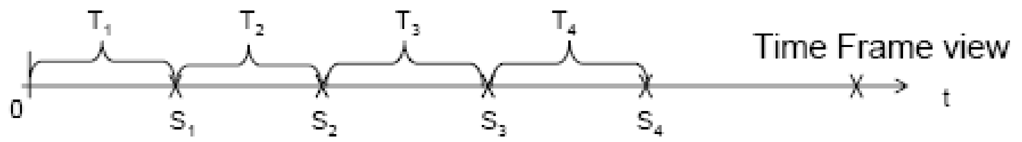
- The proposition gives us another way of defining a Poisson process.

- Suppose we have a sequence $\{T_n, n \geq 1\}$ of independent identically distributed exponential random variables, each having mean $1/\lambda$.

- Then by defining a counting process by saying that the nth event of this process occurs at time $S_n = T_1+T_2+T_3+\ldots+T_n$, the resultant counting process $\{N(t), t \geq 0\}$ will be Poisson with rate λ.

Waiting time distribution

- Another quantity of interest is S_n, the arrival time of the nth event (also called the waiting time until the nth event)



$$S_n = \sum_{i=1}^{n} T_i$$

Example

- Suppose that people immigrate into a territory at a Poisson rate $\lambda=1$ per day.

  a) What is the expected time until the tenth immigrant arrives?

  b) What is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two days?

Solution

- Since the arrival time of the nth event is Gamma distributed with parameters n, λ,

  a) The expected time until 10th arrival is, $E[S_{10}] = 10/\lambda = 10$ days

  b) From before, we know that T_{11} is exponential and independent of T_{10}, so $P\{T_{11}>2\} = e^{-2\lambda} = e^{-2} \approx .133$

Further properties of Poisson process

Property 1a:

- Consider a Poisson process $\{N(t), t \geq 0\}$ having rate λ.
- Each time an event occurs, it is classified as either a Type I or Type II event with probability p and 1-p respectively, independently of all other events.
- Let $N_1(t)$ and $N_2(t)$ denote respectively the number of Type I and Type II events occurring in [0,t].
- **Then $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are Poisson processes having respective rates λp and $\lambda(1-p)$.**
- **The two processes are also independent.**

Example

- Customers arrive at a Starbucks at a Poisson rate of $\lambda=10$ per hour.
- Suppose that each customer is a man with probability ½, and a woman with probability ½.
- Suppose you observe 100 men arrive in the first 5 hours, how many women would we expect to have arrived in that 5 hours?

Is it right??

- Because the number of male arrivals is 100, and because each arrival is male with probability ½, then the expected number of total arrivals should be 200
  - hence, the expected number of female arrivals in first 5 hours is 100.
  - Is this reasoning correct?

Absolutely not!!

- We have just shown by the previous property that the two Poisson processes {Nmale(t), $t \geq 0$} & {Nfemale(t), $t \geq 0$} are independent!
- So, the expected number of female arrivals in the first five hours is independent of the number of male arrivals in that period
- **E[Nfemale(5)] = $\lambda t(1-p)$ = (10)(5)(1/2) = 25**

Further properties of Poisson process

Property 1b:

- Let $\{N(t), t\geq 0\}$ be a Poisson process with rate λ.
- Suppose that each event is type i with probability p_i, i=1 to M, and

$$\sum_{i=1}^{M} p_i = 1$$

- Then the type i events $\{N_i(t), t\geq 0\}$ form a Poisson process with rate $p_i\lambda$; Moreover, $\{N_i(t), t\geq 0\}$ and $\{N_j(t), t\geq 0\}$ are independent of each other.

Further properties of Poisson process

Property 2:

- Let $\{N_1(t), t\geq 0\}$ and $\{N_2(t), t\geq 0\}$ be independent Poisson processes with rate λ_1 and λ_2 respectively.
- Define $N(t) = N_1(t)+N_2(t)$ for all t.
- Then $\{N(t), t\geq 0\}$ is a Poisson process with rate $\lambda=\lambda_1+\lambda_2$.

Summary

- Definition of Poisson Process
  - Definitions 1 & 2 (equivalent)
  - Counting process: Independent increments, stationary increments, Poisson distributed
- Characteristics
  - Interarrival times
    - Exponential
  - Waiting times
    - Sum of Exponential → Gamma
- Keys to obtaining these results and solving problems are
  (i) observing time frame & counting frame views
  (ii) identifying equivalent events
  (iii) stationary & independent increments
  (iv) properties of distributions

Summary

- Properties
  - If a PP $\{N(t), t \geq 0\}$ has rate λ,
    - If each event is type I or type II with probability p & (1-p), then $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are **independent** PP having respective rates λp and $\lambda(1-p)$.
    - If each event is type i with probability p_i, i=1 to M, and $\sum_{i=1}^{M} p_i = 1$ then the type i events $\{N_i(t), t \geq 0\}$ form independent PPs with rate $p_i\lambda$
  - If $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent PPs with rate λ_1 and λ_2, and $N(t) = N_1(t)+N_2(t)$ for all t, then $\{N(t), t \geq 0\}$ is a PP with rate $\lambda=\lambda_1+\lambda_2$.