



BIJU PATNAIK UNIVERSITY OF TECHNOLOGY,  
ODISHA

Lecture Notes

On

**Markov Chain**

**Part 1**

Prepared by,  
Dr. Subhendu Kumar Rath,  
BPUT, Odisha.



# Markov Chain Part 1

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Dr. Subhendu Kumar Rath  
Deputy Registrar, BPUT



# Outline

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- Stochastic Process
- Markov Chain



# Stochastic Process

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- A stochastic process is a indexed collection of random variables  $\{X_t\} = \{ X_0, X_1, X_2, \dots \}$  for describing the behavior of a system operating over some period of time.
- For example :
  - $X_0 = 3, X_1 = 2, X_2 = 1, X_3 = 0, X_4 = 3, X_5 = 1$



## Stochastic Process (cont.)

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- An inventory example:
- A camera store stocks a particular model camera.
- $D_t$  represents the demand for this camera during week  $t$ .
- $D_t$  has a Poisson distribution with a mean of 1.
- $X_t$  represents the number of cameras on hand at the end of week  $t$ . ( $X_0 = 3$ )
- If there are no cameras in stock on Saturday night, the store orders three cameras.
- $\{ X_t \}$  is a stochastic process.
- $$X_{t+1} = \begin{cases} \max\{ 3 - D_{t+1}, 0 \} & \text{if } X_t = 0 \\ \max\{ X_t - D_{t+1}, 0 \} & \text{if } X_t \geq 1 \end{cases}$$



# Markov Chain

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- A stochastic process  $\{X_t\}$  is a Markov chain if it has Markovian property.
- Markovian property:
  - $P\{ X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \dots, X_{t-1} = k_{t-1}, X_t = i \}$   
 $= P\{ X_{t+1} = j \mid X_t = i \}$
- $P\{ X_{t+1} = j \mid X_t = i \}$  is called the transition probability.



## Markov Chain (con.)

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- Stationary transition probability:
  - If ,for each  $i$  and  $j$ ,  $P\{ X_{t+1} = j \mid X_t = i \} = P\{ X_1 = j \mid X_0 = i \}$ , for all  $t$ , then the transition probability are said to be stationary.



# Markov Chain (con.)

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- Formulating the inventory example:
  - Transition matrix:

$$\mathbf{P} = \begin{array}{c|cccc} & \text{state } 0 & 1 & 2 & 3 \\ \hline 0 & p_{00} & p_{01} & p_{02} & p_{03} \\ 1 & p_{10} & p_{11} & p_{12} & p_{13} \\ 2 & p_{20} & p_{21} & p_{22} & p_{23} \\ 3 & p_{30} & p_{31} & p_{32} & p_{33} \end{array}$$





## Markov Chain (con.)

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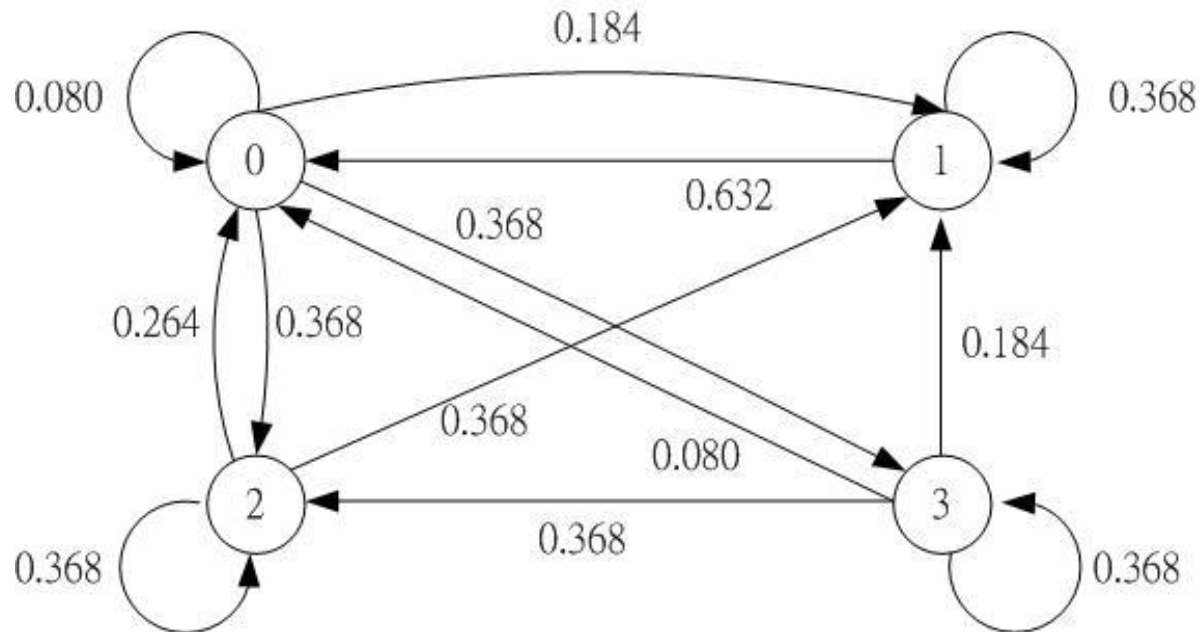
- $$X_{t+1} = \begin{cases} \max\{ 3 - D_{t+1}, 0 \} & \text{if } X_t = 0 \\ \max\{ X_t - D_{t+1}, 0 \} & \text{if } X_t \geq 1 \end{cases}$$

- $p_{03} = P\{ D_{t+1} = 0 \} = 0.368$
- $p_{02} = P\{ D_{t+1} = 1 \} = 0.368$
- $p_{01} = P\{ D_{t+1} = 2 \} = 0.184$
- $p_{00} = P\{ D_{t+1} \geq 3 \} = 0.080$

	state	0	1	2	3
<b>P</b> =	0	0.080	0.184	0.368	0.368
	1	0.632	0.368	0.000	0.000
	2	0.264	0.368	0.368	0.000
	3	0.080	0.184	0.368	0.368

# Markov Chain (con.)

- The state transition diagram:





## Markov Chain (con.)

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- n-step transition probability :

- $p_{ij}^{(n)} = P\{ X_{t+n} = j \mid X_t = i \}$

- n-step transition matrix :

$$\mathbf{P}^{(n)} = \begin{array}{ccccc} & \text{state } 0 & 1 & \dots & M \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ M \end{array} & \begin{array}{c} P_{00}^{(n)} \\ P_{10}^{(n)} \\ \dots \\ P_{M0}^{(n)} \end{array} & \begin{array}{c} P_{01}^{(n)} \\ P_{11}^{(n)} \\ \dots \\ P_{M1}^{(n)} \end{array} & \begin{array}{c} \dots \\ \dots \\ \dots \\ \dots \end{array} & \begin{array}{c} P_{0M}^{(n)} \\ P_{1M}^{(n)} \\ \dots \\ P_{MM}^{(n)} \end{array} \end{array}$$



## Markov Chain (con.)

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- Chapman-Kolmogorove Equation :

$$P_{ij}^{(n)} = \sum_{k=0}^M P_{ik}^{(m)} P_{kj}^{(n-m)} \quad \begin{array}{l} \text{for all } i = 0, 1, \dots, M, \\ \text{ } j = 0, 1, \dots, M, \\ \text{and any } m = 1, 2, \dots, n-1, \\ \text{ } n = m+1, m+2, \dots \end{array}$$

- The special cases of  $m = 1$  leads to :

$$P_{ij}^{(n)} = \sum_{k=0}^M P_{ik}^{(1)} P_{kj}^{(n-1)} \quad \text{for all } i \text{ and } j$$

- Thus the  $n$ -step transition probability can be obtained from one-step transition probability recursively.



## Markov Chain (con.)

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- Conclusion :

- $\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}\mathbf{P}\mathbf{P}^{(n-2)} = \dots = \mathbf{P}^n$

- n-step transition matrix for the inventory example :

	state 0	1	2	3
0	0.080	0.184	0.368	0.368
1	0.632	0.368	0.000	0.000
2	0.264	0.368	0.368	0.000
3	0.080	0.184	0.368	0.368

$\mathbf{P} =$

	state 0	1	2	3
0	0.289	0.286	0.261	0.164
1	0.282	0.285	0.268	0.166
2	0.284	0.283	0.263	0.171
3	0.289	0.286	0.261	0.164

$\mathbf{P}^{(4)} =$



## Markov Chain (con.)

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- What is the probability that the camera store will have three cameras on hand 4 weeks after the inventory system began ?
- $$P\{ X_n = j \} = P\{ X_0 = 0 \} p_{0j}^{(n)} + P\{ X_0 = 1 \} p_{1j}^{(n)} + \dots$$
$$+ P\{ X_0 = M \} p_{Mj}^{(n)}$$
- $$P\{ X_4 = 3 \} = P\{ X_0 = 0 \} p_{03}^{(4)} + P\{ X_0 = 1 \} p_{13}^{(4)}$$
$$+ P\{ X_0 = 2 \} p_{23}^{(4)} + P\{ X_0 = 3 \} p_{33}^{(4)}$$
$$= (1) p_{33}^{(4)} = 0.164$$



## Markov Chain (con.)

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- Long-Run Properties of Markov Chain
  - Steady-State Probability

	state 0	1	2	3
0	0.080	0.184	0.368	0.368
<b>P</b> = 1	0.632	0.368	0.000	0.000
2	0.264	0.368	0.368	0.000
3	0.080	0.184	0.368	0.368

	state 0	1	2	3
0	0.286	0.285	0.264	0.166
<b>P</b> (8) = 1	0.286	0.285	0.264	0.166
2	0.286	0.285	0.264	0.166
3	0.286	0.285	0.264	0.166



## Markov Chain (con.)

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- The steady-state probability implies that there is a limiting probability that the system will be in each state  $j$  after a large number of transitions, and that this probability is independent of the initial state.
- Not all Markov chains have this property.

state	0	1	2	3
0	$\pi_0$	$\pi_1$	$\pi_2$	$\pi_3$
1	$\pi_0$	$\pi_1$	$\pi_2$	$\pi_3$
2	$\pi_0$	$\pi_1$	$\pi_2$	$\pi_3$
3	$\pi_0$	$\pi_1$	$\pi_2$	$\pi_3$





## Markov Chain (con.)

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- Steady-State Equations :

$$\pi_j = \sum_{i=0}^M \pi_i p_{ij} \quad \text{for } i = 0, 1, \dots, M$$

$$\sum_{j=0}^M \pi_j = 1$$

- , which consists of  $M+2$  equations in  $M+1$  unknowns.



## Markov Chain (con.)

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- The inventory example :
- $\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30}$  ,
- $\pi_1 = \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31}$  ,
- $\pi_2 = \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32}$  ,
- $\pi_3 = \pi_0 p_{03} + \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33}$  ,
- $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$ .
  
- $\pi_0 = 0.080\pi_0 + 0.632\pi_1 + 0.264\pi_2 + 0.080\pi_3$  ,
- $\pi_1 = 0.184\pi_0 + 0.368\pi_1 + 0.368\pi_2 + 0.184\pi_3$  ,
- $\pi_2 = 0.368\pi_0 + \quad \quad \quad + 0.368\pi_2 + 0.368\pi_3$  ,
- $\pi_3 = 0.368\pi_0 + \quad \quad \quad + \quad \quad \quad + 0.368\pi_3$  ,
- $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$ .
  
- $\pi_0 = 0.286, \pi_1 = 0.285, \pi_2 = 0.263, \pi_3 = 0.166$



# Reference

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- *Hillier and Lieberman*, “Introduction to Operations Research”, seventh edition, McGraw Hill



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- THANK YOU

Q & A