BIJU PATNAIK UNIVERSITY OF TECHNOLOGY, ODISHA

Lecture Notes

On

DIFFERENCE EQUATION AND ITS APPLICATION

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DIFFERENCE EQUATION AND ITS APPLICATION

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DEFINITION

An expression which expresses a relation between an independent variable and successive values or

Successive differences of dependent variable is called a difference equation.

EXAMPLES

- $y_{x+2} - 3y_{x+1} + 2y_x = x^2$
- $\Delta^2 y_x - 2\Delta y_x - 4y_x = 0$
- $\Delta^3 y_x - 3\Delta^2 y_x - 2\Delta y_x - 4y_x = x(x-1)$

ORDER AND DEGREE

- The order of a difference equation expressed in terms of successive values of y is the difference between the highest and lowest subscripts or arguments of y
- The degree of a difference equation free from Δ' s is the highest exponent of the y's.

SOLUTION

- A solution of a difference equation is any function that satisfies it.
- The general solution of a difference equation of order n is a solution that contains n arbitrary constants or n arbitrary function which are periodic with period equal to the interval of differencing.

CONTD…..

- The particular solution of a difference equation is a solution obtained by assigning particular values to the arbitrary constants or functions.

LINEAR DIFFERENCE EQATION

- A difference equation in which y_x, y_{x+1}, y_{x+2}……… occur in the first degree and are not multiplied together is said to be linear difference equation.
- The general form of a linear difference equation is as follows

CONTD….

- $a_0 y_{x+n} + a_1 y_{x+n-1} + \dots\dots\dots\ a_{n-1} y_{x+1} + a_n y_x$ = R(x)

Where a_i and R(x) are known function of x.

- If R(x) is zero then the above equation is called homogeneous difference equation.
- The solution can be obtained by the relation $y_x = u_x + v_x$, where u_x is called complementary function and v_x uis called particular integral.

APPLICATION

- Difference equation can be applicable in the following areas.
  - Numerical methods to solve partial differential equation.
  - Fourier series
  - Algebra and Analysis

First Derivative Approximations

- Backward difference: $(u_j - u_{j-1}) / \Delta x$
- Forward difference: $(u_{j+1} - u_j) / \Delta x$
- Centered difference: $(u_{j+1} - u_{j-1}) / 2\Delta x$

Taylor Expansion

- $u(x + \Delta x) = u(x) + u'(x)\Delta x + 1/2\, u''(x)(\Delta x)^2$
  $+ 1/6\, u'''(x)(\Delta x)^3 + O(\Delta x)^4$

- $u(x - \Delta x) = u(x) - u'(x)\Delta x + 1/2\, u''(x)(\Delta x)^2$
  $- 1/6\, u'''(x)(\Delta x)^3 + O(\Delta x)^4$

Taylor Expansion

$$u'(x) = \frac{u(x) - u(x - \Delta x)}{\Delta x} + O(\Delta x)$$

$$u'(x) = \frac{u(x + \Delta x) - u(x)}{\Delta x} + O(\Delta x)$$

$$u'(x) = \frac{u(x + \Delta x) - u(x - \Delta x)}{2\Delta x} + O(\Delta x)^2$$

Second Derivative Approximation

- Centered difference: $(u_{j+1} - 2u_j + u_{j-1}) / (\Delta x)^2$
- Taylor Expansion

$$u''(x) = \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{(\Delta x)^2} + O(\Delta x)^2$$

Function of Two Variables

$$u(j\Delta x, n\Delta t) \sim u_j^n$$

- Backward difference for *t* and *x*

$$\frac{\partial u}{\partial t}(j\Delta x, n\Delta t) \sim (u_j^n - u_j^{n-1}) / \Delta t$$

$$\frac{\partial u}{\partial x}(j\Delta x, n\Delta t) \sim (u_j^n - u_j^{n-1}) / \Delta x$$

Function of Two Variables

- Forward difference for t and x

$$\frac{\partial u}{\partial t}(j\Delta x, n\Delta t) \sim (u_j^{n+1} - u_j^n) / \Delta t$$

$$\frac{\partial u}{\partial x}(j\Delta x, n\Delta t) \sim (u_j^{n+1} - u_j^n) / \Delta x$$

Function of Two Variables

- Centered difference for *t* and *x*

$$\frac{\partial u}{\partial t}(j\Delta x, n\Delta t) \sim (u_j^{n+1} - u_j^{n-1}) / (2\Delta t)$$

$$\frac{\partial u}{\partial x}(j\Delta x, n\Delta t) \sim (u_j^{n+1} - u_j^{n-1}) / (2\Delta x)$$

Partial Differential Equations

- Partial Differential Equations (PDEs).
- What is a PDE?
- Examples of Important PDEs.
- Classification of PDEs.

Partial Differential Equations

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

Example:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

PDE involves two or more independent variables

(in the example x and t are independent variables)

Notation

$$u_{xx} = \frac{\partial^2 \ u(x,t)}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2 \ u(x,t)}{\partial x \ \partial t}$$

Order of the PDE = order of the highest order derivative.

Examples of PDEs

PDEs are used to model many systems in many different fields of science and engineering.

Important Examples:

- Laplace Equation
- Heat Equation
- Wave Equation

Laplace Equation

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

Used to describe the steady state distribution of heat in a body.

Also used to describe the steady state distribution of electrical charge in a body.

Heat Equation

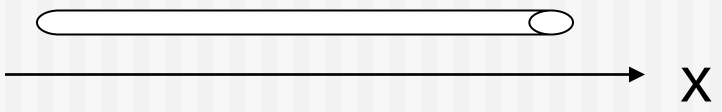
$$\frac{\partial u(x,y,z,t)}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

The function $u(x,y,z,t)$ is used to represent the temperature at time t in a physical body at a point with coordinates (x,y,z)

α is the thermal diffusivity. It is sufficient to consider the case $\alpha = 1$.

Simpler Heat Equation

$$\frac{\partial\, T(x,t)}{\partial t} = \frac{\partial^2\, T(x,t)}{\partial x^2}$$



T(x,t) is used to represent the temperature at time *t* at the point *x* of the thin rod.

Wave Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

The function *u(x,y,z,t)* is used to represent the displacement at time t of a particle whose position at rest is *(x,y,z)* .

The constant *c* represents the propagation speed of the wave.

Classification of PDEs

Linear Second order PDEs are important sets of equations that are used to model many systems in many different fields of science and engineering.

Classification is important because:

- Each category relates to specific engineering problems.
- Different approaches are used to solve these categories.

Linear Second Order PDEs
Classification

A second order linear PDE (2 - independent variables)

$$A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of x, y, u, u_x, and u_y

is classified based on $(B^2 - 4AC)$ as follows:

$B^2 - 4AC < 0$	Elliptic
$B^2 - 4AC = 0$	Parabolic
$B^2 - 4AC > 0$	Hyperbolic

Linear Second Order PDE
Examples (Classification)

$$\text{Laplace Equation} \quad \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1 \Rightarrow B^2 - 4AC < 0$$

$$\Rightarrow \text{Laplace Equation } \mathit{is\ Elliptic}$$

$$\text{One possible solution: } u(x,y) = e^x \sin y$$

$$u_x = e^x \sin y, \quad u_{xx} = e^x \sin y$$

$$u_y = e^x \cos y, \quad u_{yy} = -e^x \sin y$$

$$u_{xx} + u_{yy} = 0$$

Linear Second Order PDE
Examples (Classification)

$$\text{Heat Equation} \quad \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$A = \alpha, \ B = 0, \ C = 0 \Rightarrow B^2 - 4AC = 0$$

$$\Rightarrow \text{Heat Equation } is\ Parabolic$$

$$\text{Wave Equation} \quad c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

$$A = c^2 > 0, \ B = 0, \ C = -1 \Rightarrow B^2 - 4AC > 0$$

$$\Rightarrow \text{Wave Equation } is\ Hyperbolic$$

Boundary Conditions for PDEs

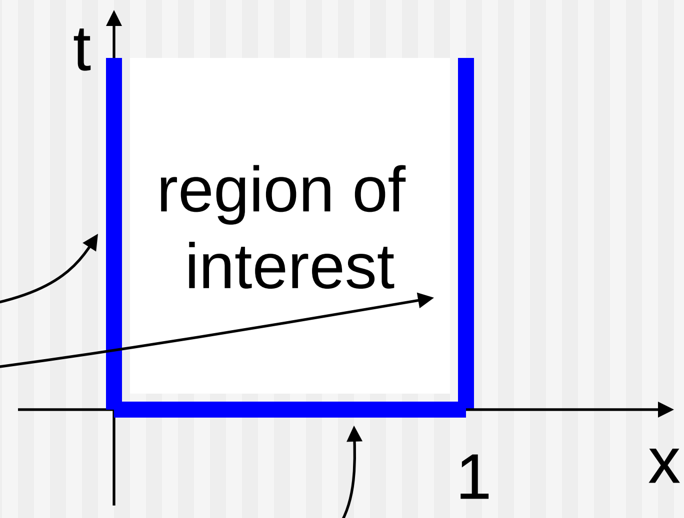
- To uniquely specify a solution to the PDE, a set of boundary conditions are needed.
- Both regular and irregular boundaries are possible.

$$\text{Heat Equation}: \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



The Solution Methods for PDEs

- Analytic solutions are possible for simple and special (idealized) cases only.
- To make use of the nature of the equations, different methods are used to solve different classes of PDEs.
- The methods discussed here are based on the finite difference technique.

Parabolic Equations

- Parabolic Equations
- Heat Conduction Equation
- Explicit Method
- Implicit Method
- Cranks Nicolson Method

Parabolic Equations

A second order linear PDE (2 - independent variables x, y)

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D = 0,$$

A, B, and C are functions of x and y

D is a function of x, y, u, u_x, and u_y

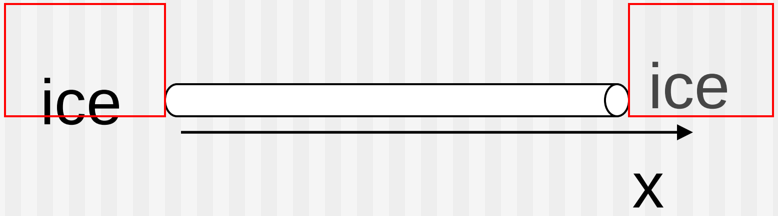
is parabolic if $$B^2 - 4AC = 0$$

Parabolic Problems

Heat Equation : $\dfrac{\partial\, T(x,t)}{\partial t} = \dfrac{\partial^2\, T(x,t)}{\partial x^2}$

$T(0,t) = T(1,t) = 0$

$T(x,0) = \sin(\pi\, x)$

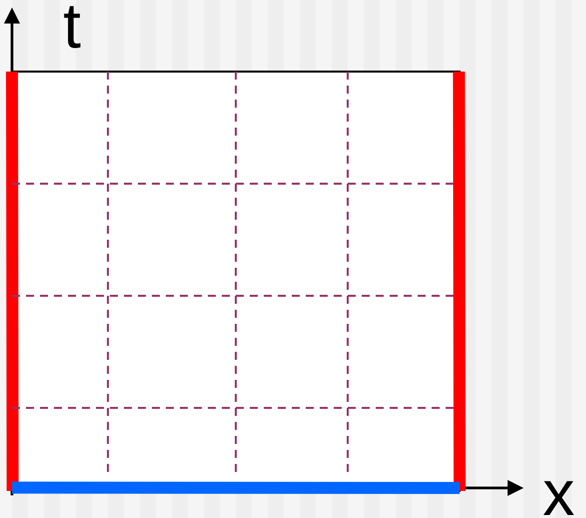


* Parabolic problem $(B^2 - 4AC = 0)$

* Boundary conditions are needed to uniquely specify a solution.

Finite Difference Methods

- Divide the interval *x* into sub-intervals, each of width *h*
- Divide the interval *t* into sub-intervals, each of width *k*
- A grid of points is used for the finite difference solution
- $T_{i,j}$ represents $T(x_i, t_j)$
- Replace the derivates by finite-difference formulas



Finite Difference Methods

Replace the derivatives by finite difference formulas

Central Difference Formula for $\dfrac{\partial^2 T}{\partial x^2}$:

$$\frac{\partial^2 T(x,t)}{\partial x^2} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2}$$

Forward Difference Formula for $\dfrac{\partial T}{\partial t}$:

$$\frac{\partial T(x,t)}{\partial t} \approx \frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{T_{i,j+1} - T_{i,j}}{k}$$

Solution of the Heat Equation

- Two solutions to the Parabolic Equation (Heat Equation) will be presented:

1. Explicit Method:

   Simple, Stability Problems.

2. Crank-Nicolson Method:

   Involves the solution of a Tridiagonal system of equations, Stable.

Explicit Method

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

$$\frac{T(x,t+k) - T(x,t)}{k} = \frac{T(x-h,t) - 2T(x,t) + T(x+h,t)}{h^2}$$

$$T(x,t+k) - T(x,t) = \frac{k}{h^2}\left(T(x-h,t) - 2T(x,t) + T(x+h,t)\right)$$

$$\text{Define} \quad \lambda = \frac{k}{h^2}$$

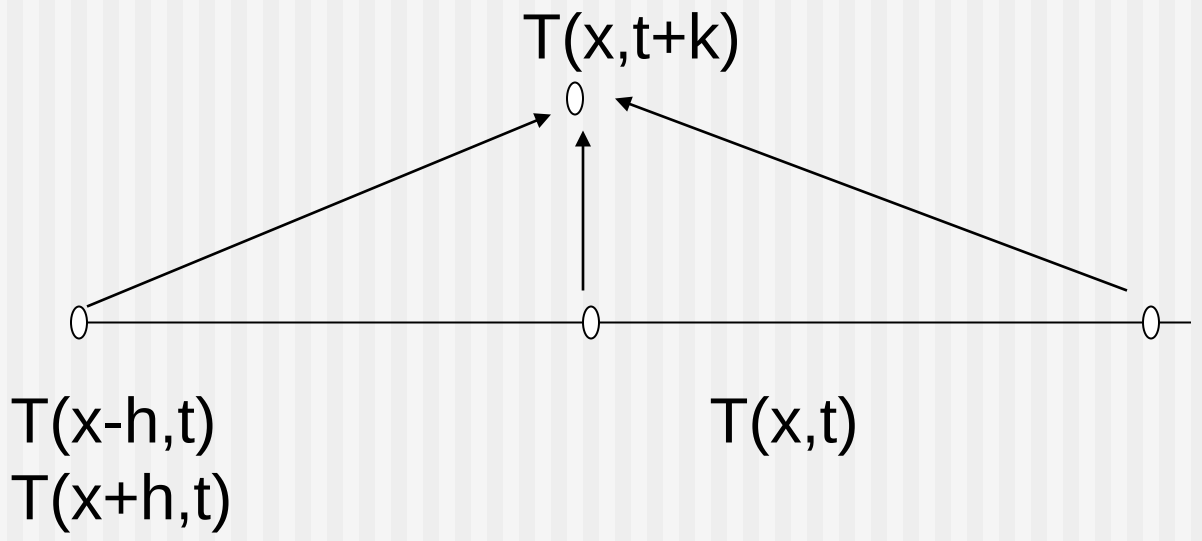
$$T(x,t+k) = \lambda\, T(x-h,t) + (1-2\lambda)\, T(x,t) + \lambda\, T(x+h,t)$$

Explicit Method

How Do We Compute?

$$T(x,t+k) = \lambda\ T(x-h,t) + (1-2\lambda)\ T(x,t) + \lambda\ T(x+h,t)$$

means



Convergence and Stability

$T(x,t+k)$ can be computed directly using :

$$T(x,t+k) = \lambda\, T(x-h,t) + (1-2\lambda)\, T(x,t) + \lambda\, T(x+h,t)$$

Can be unstable (errors are magnified)

To guarantee stability, $(1-2\lambda) \geq 0 \Rightarrow \lambda \leq \dfrac{1}{2} \Rightarrow k \leq \dfrac{h^2}{2}$

This means that k is much smaller than h

This makes it slow.

Example 1

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

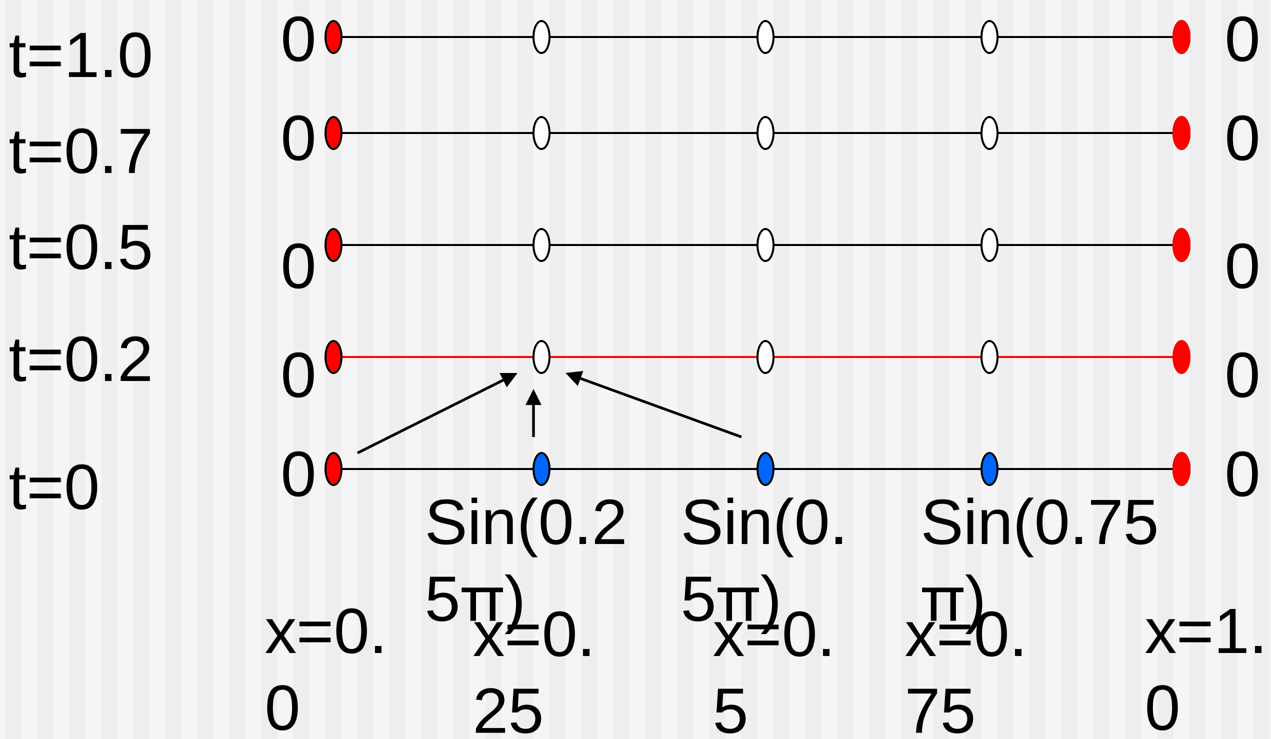
$$\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} - \frac{u(x,t+k) - u(x,t)}{k} = 0$$

$$16\big(u(x-h,t) - 2u(x,t) + u(x+h,t)\big) - 4\big(u(x,t+k) - u(x,t)\big) = 0$$

$$u(x,t+k) = 4\ u(x-h,t) - 7\ u(x,t) + 4\ u(x+h,t)$$

Example 1

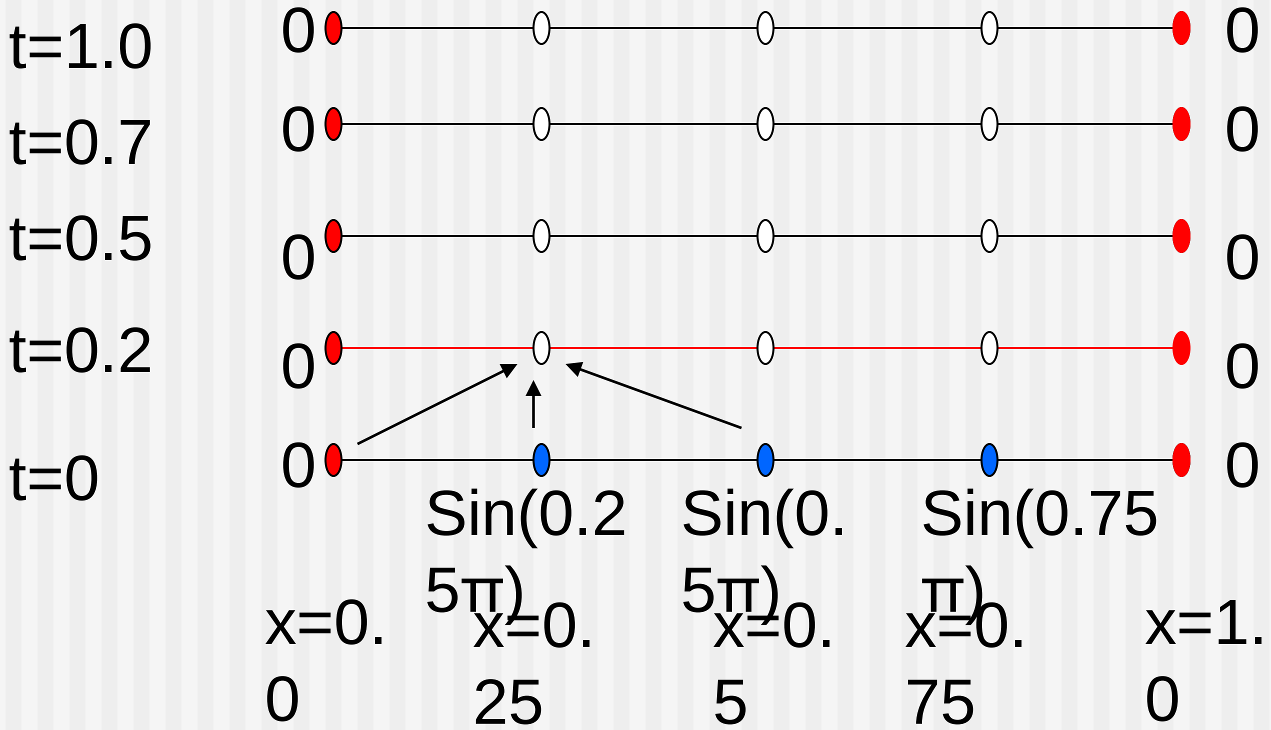
$$u(x,t+k) = 4\ u(x-h,t) - 7\ u(x,t) + 4\ u(x+h,t)$$



Example 1

$$u(0.25,0.25) = 4\ u(0,0) - 7\ u(0.25,0) + 4\ u(0.5,0)$$

$$= 0 - 7\sin(\pi/4) + 4\sin(\pi/2) = -0.9497$$



Crank-Nicolson Method

The method involves solving a Tridiagonal system of linear equations.

The method is stable (No magnification of error).

$\rightarrow$ We can use larger h, k (compared to the Explicit Method).

Crank-Nicolson Method

Based on the finite difference method

1. Divide the interval x into subintervals of width h

2. Divide the interval t into subintervals of width k

3. Replace the first and second partial derivatives with their *backward* and *central difference* formulas respectively :

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t) - u(x,t-k)}{k}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2}$$

Crank-Nicolson Method

Heat Equation : $\dfrac{\partial^2 \; u(x,t)}{\partial x^2} = \dfrac{\partial \; u(x,t)}{\partial t}$ becomes

$$\frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} = \frac{u(x,t)-u(x,t-k)}{k}$$

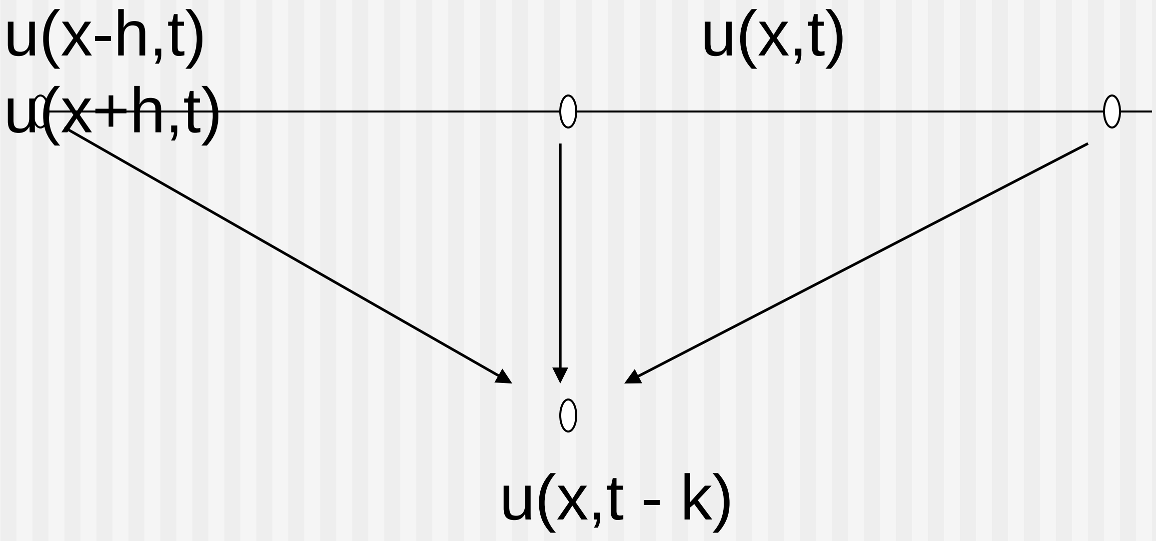
$$\frac{k}{h^2}\left(u(x-h,t)-2u(x,t)+u(x+h,t)\right) = u(x,t)-u(x,t-k)$$

$$-\frac{k}{h^2}u(x-h,t)+(1+2\frac{k}{h^2})\,u(x,t)-\frac{k}{h^2}u(x+h,t) = u(x,t-k)$$

Crank-Nicolson Method

Define $\lambda = \dfrac{k}{h^2}$ then Heat equation becomes:

$$-\lambda\, u(x-h,t) + (1+2\lambda)\, u(x,t) - \lambda\, u(x+h,t) = u(x,t-k)$$



Crank-Nicolson Method

The equation :

$$-\lambda\, u(x-h,t)+(1+2\lambda)\, u(x,t)-\lambda\, u(x+h,t)=u(x,t-k)$$

can be rewritten as :

$$-\lambda\, u_{i-1,j}+(1+2\lambda)\, u_{i,j}-\lambda\, u_{i+1,j}=u_{i,j-1}$$

and can be expanded as a system of equations (fix $j=1$) :

$$-\lambda\, u_{0,1}+(1+2\lambda)\, u_{1,1}-\lambda\, u_{2,1}=u_{1,0}$$

$$-\lambda\, u_{1,1}+(1+2\lambda)\, u_{2,1}-\lambda\, u_{3,1}=u_{2,0}$$

$$-\lambda\, u_{2,1}+(1+2\lambda)\, u_{3,1}-\lambda\, u_{4,1}=u_{3,0}$$

$$-\lambda\, u_{3,1}+(1+2\lambda)\, u_{4,1}-\lambda\, u_{5,1}=u_{4,0}$$

Crank-Nicolson Method

$$-\lambda u(x-h,t)+(1+2\lambda)\,u(x,t)-\lambda u(x+h,t)=u(x,t-k)$$

can be expressed as a Tridiagonal system of equations :

$$\begin{bmatrix} 1+2\lambda & -\lambda & & \\ -\lambda & 1+2\lambda & -\lambda & \\ & -\lambda & 1+2\lambda & -\lambda \\ & & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} = \begin{bmatrix} u_{1,0}+\lambda\, u_{0,1} \\ u_{2,0} \\ u_{3,0} \\ u_{4,0}+\lambda\, u_{5,1} \end{bmatrix}$$

where $u_{1,0}, u_{2,0}, u_{3,0}$, and $u_{4,0}$ are the initial temperature values

at $x = x_0 + h,\; x_0 + 2h,\; x_0 + 3h,$ and $x_0 + 4h$

$u_{0,1}$ and $u_{5,1}$ are the boundary values at $x = x_0$ and $x_0 + 5h$

Crank-Nicolson Method

The solution of the tridiagonal system produces:

The temperature values $u_{1,1}, u_{2,1}, u_{3,1}$, and $u_{4,1}$ at $t = t_0 + k$

To compute the temperature values at $t = t_0 + 2k$

Solve a second tridiagonal system of equations ($j = 2$)

$$\begin{bmatrix} 1+2\lambda & -\lambda & & \\ -\lambda & 1+2\lambda & -\lambda & \\ & -\lambda & 1+2\lambda & -\lambda \\ & & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} u_{1,1} + \lambda\, u_{0,2} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} + \lambda\, u_{5,2} \end{bmatrix}$$

To compute $u_{1,2}, u_{2,2}, u_{3,2}$, and $u_{4,2}$

Repeat the above step to compute temperature values at $t_0 + 3k$, etc.

Example 2

Solve the PDE :

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi\, x)$$

Solve using Crank - Nicolson method

$Use \quad h = 0.25, \quad k = 0.25$ to find $u(x,t)$ for $x \in [0,1], t \in [0,1]$

Example 2

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$16\big(u(x-h,t) - 2u(x,t) + u(x+h,t)\big) - 4\big(u(x,t) - u(x,t-k)\big) = 0$$

$$Define \quad \lambda = \frac{k}{h^2} = 4$$

$$-4\,u(x-h,t) + 9\,u(x,t) - 4\,u(x+h,t) = u(x,t-k)$$

$$-4\,u_{i-1,j} + 9\,u_{i,j} - 4\,u_{i+1,j} = u_{i,j-1}$$

Example 2

$$-4u_{0,1} + 9u_{1,1} - 4u_{2,1} = u_{1,0} \Rightarrow \quad 9u_{1,1} - 4u_{2,1} = \sin(\pi/4)$$

$$-4u_{1,1} + 9u_{2,1} - 4u_{3,1} = u_{2,0} \Rightarrow -4u_{1,1} + 9u_{2,1} - 4u_{3,1} = \sin(\pi/2)$$

$$-4u_{2,1} + 9u_{3,1} - 4u_{4,1} = u_{3,0} \Rightarrow \quad -4u_{2,1} + 9u_{3,1} = \sin(3\pi/4)$$

Example 2

Solution of Row 1 at t1=0.25 sec

The Solution of the PDE at $t_1 = 0.25$ sec is the solution of the following tridiagonal system of equations:

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} \sin(0.25\pi) \\ \sin(0.5\pi) \\ \sin(0.75\pi) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 0.21151 \\ 0.29912 \\ 0.21151 \end{bmatrix}$$

Example 2:

Second Row at t2=0.5 sec

$$-4u_{0,2}+9u_{1,2}-4u_{2,2}=u_{1,1}\Rightarrow \quad 9u_{1,2}-4u_{2,2} \qquad =0.21151$$

$$-4u_{1,2}+9u_{2,2}-4u_{3,2}=u_{2,1}\Rightarrow -4u_{1,2}+9u_{2,2}-4u_{3,2}=0.29912$$

$$-4u_{2,2}+9u_{3,2}-4u_{4,2}=u_{3,1}\Rightarrow \qquad -4u_{2,2}+9u_{3,2}=0.21151$$

Example 2

Solution of Row 2 at t2=0.5 sec

The Solution of the PDE at $t_2 = 0.5$ sec is the solution of the following tridiagonal system of equations:

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 0.21151 \\ 0.29912 \\ 0.21151 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} 0.063267 \\ 0.089473 \\ 0.063267 \end{bmatrix}$$

Example 2

Solution of Row 3 at t3=0.75 sec

The Solution of the PDE at $t_3 = 0.75$ sec is the solution of the following tridiagonal system of equations :

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} 0.063267 \\ 0.089473 \\ 0.063267 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} 0.018924 \\ 0.026763 \\ 0.018924 \end{bmatrix}$$

Example 2

Solution of Row 4 at t4=1 sec

The Solution of the PDE at $t_4 = 1\,\text{sec}$ is the solution of the following tridiagonal system of equations:

$$\begin{bmatrix} 9 & -4 & \\ -4 & 9 & -4 \\ & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \end{bmatrix} = \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} 0.018924 \\ 0.026763 \\ 0.018924 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \end{bmatrix} = \begin{bmatrix} 0.0056606 \\ 0.0080053 \\ 0.0056606 \end{bmatrix}$$

THANK YOU